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THIS JOURNAL IS DEDICATED TO THE FOLLOWING AIMS: (1) Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values. (2) To supply an additional medium for the publication of expository Mathematical articles. (3) To promote more scientific methods of teaching mathematics. (4) To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Every paper on technical mathematics offered for publication should be submitted (with enough enclosed postage to cover two two-way transmissions) to the Chairman of the appropriate Committee, or to a Committee member whom the Chairman may designate to examine it, after being requested to do so by the writer. If approved for publication, the Committee will forward it to the Editor and Manager at Baton Rouge, who will notify the writer of its acceptance for publication. If the paper is not approved the Committee will so notify the Editor and Manager, who will inform the writer accordingly.

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A Sweeping Publicity Program for the Magazine

In a few weeks a circular describing all phases of the NATIONAL MATHEMATICS MAGAZINE and its work will be mailed from Baton Rouge to 6,000 mathematics teachers in high schools, junior colleges, colleges, and universities in the United States and Canada. Five thousand of the names and addresses for this colossal undertaking have been supplied by Secretary Kline of the American Mathematical Society, and one thousand by former president, Mary Potter, of the National Council of Teachers of Mathematics.

A letter sent out early in May to fifty leaders in American Mathematical organizations emphasized the importance of a cooperative support for the MAGAZINE by all whose life-work is in the field of mathematics. This letter bore the following statement: "While NATIONAL MATHEMATICS MAGAZINE has no official relation to existing mathematical bodies, such as the Society, the Association, or the National Council of Teachers of Mathematics, none-the-less it can be said emphatically that it shares with other journals a sense of responsibility for the advancement of mathematical programs on a national scale."

The very first answer to our letter to fifty has come from Marston Morse, President of the American Mathematical Society, and topmost figure in the American program of an all-out mathematical aid in the cause of National Defense. We quote his letter:

MARSTON MORSE, President
American Mathematical Society, The Institute for Advanced Study
PRINCETON, NEW JERSEY

"Dear Sanders:

"I wholeheartedly approve of your endeavor to keep the NATIONAL MATHEMATICS MAGAZINE going in an effective way. This is very important now, and will be even more important after the war. The appeal should be broadened as far as possible, including the cultural influence of mathematics and its relation to economics and philosophy. Please accept my very best wishes for your continued success."

Very truly yours,
[Signed] MARSTON MORSE.

The circular to six thousand will reach each of our subscribers in addition to thousands who at the present are not subscribers to NATIONAL MATHEMATICS MAGAZINE. We earnestly request their utmost cooperation in this program—a program aiming to make the MAGAZINE familiar to the entire mathematical world in the United States and Canada.

S. T. SANDERS,

On the Cubic of Tschirnhausen

By J. H. Weaver
Ohio State University

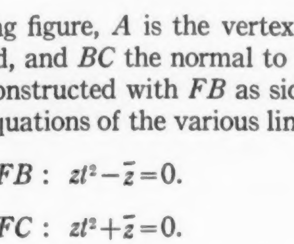
Introduction: In 1832 E. Catalan gave a new construction for a cubic already discussed by Tschirnhausen (1690) and L'Hospital (1696).[1,2]* Catalan proceeded as follows: Consider a parabola and let its focus be F. Draw a focal chord FB. At B draw the normal to the parabola. Complete the rectangle with FB as one side and the normal at B as the diagonal. Then the vertex of the rectangle, D, opposite F lies on the cubic. We propose to develop some relations existing between certain lines and parabolas associated with this curve.

The equation of the cubic. We may write the equation of the parabola in the form

$$z=\frac{1}{(1-t)^2}.\text{[3]} \tag{1}$$

The focus of the parabola is then given by $z=0$ and its axis has the equation

$$z-\bar{z}=0. \tag{2}$$



If, in the adjoining figure, A is the vertex of the parabola, F its focus, FB a focal chord, and BC the normal to the parabola at B, and $FBDC$ the rectangle constructed with FB as side and BC as diagonal, the following are the equations of the various lines and points involved.

$$FB: \ zt^2-\bar{z}=0. \tag{3}$$

$$FC: \ zt^2+\bar{z}=0. \tag{4}$$

*The numbers in brackets refer to references at end of the article.

$$BC: \quad z t+\bar{z}=\frac{t(1+t)}{(1-t)^{2}}. \tag{5}$$

$$BD: \quad z t^{2}-\bar{z}=\frac{2 t^{2}}{(1-t)^{2}}. \tag{6}$$

The lines (4) and (5) intersect in the point C which is given by the equation

$$z=\frac{1+t}{(1-t)^{3}}. \tag{7}$$

Since the coordinate of the point D is given by the sum of the coordinates of B and C we have from Equations (1) and (7) the coordinate of D which is

$$z=\frac{2}{(1-t)^{3}}. \tag{8}$$

Hence

$$FD: \quad z t^{3}+\bar{z}=0. \tag{9}$$

If we take the partial derivative of (6) with respect to t we obtain (8). *Hence the envelope of the line BD is the cubic* (8).

Properties of the curve. The clinant of the line FB is $1/t^2$, that of FD is $-1/t^3$, and that of AF is unity. Hence the angle BFA is two-thirds of the angle DFA.[4] This property enables one to use curve to trisect an angle. For this reason the curve is sometimes called the trisectrix of Catalan.

Choose the point on the cubic (8) given by $t=t_1$. Then at this point we have the parabola

$$z=\frac{2}{(1-t_{1})(1-t)^{2}}. \tag{10}$$

The following properties of the parabola (10) are easily obtained:

(a) It is tangent to the cubic (8) at the point where $t=t_1$.

(b) Its focus is at the origin.

(c) Its vertex is given by

$$z=\frac{1}{2(1-t_{1})}. \tag{11}$$

(d) The equation of its directrix is

$$zt_1 - \bar{z} = \frac{2t_1}{1-t_1}. \tag{12}$$

If in (11) we consider t_1 as a variable parameter, the vertex describes the line

$$z + \bar{z} = \tfrac{1}{2}. \tag{13}$$

This line is tangent to the cubic (8) at the point where $t = -1$.

If we take the partial derivative of (12) with respect to t_1 we obtain the envelope of the directrix of the parabola (10) for variable t_1. It is

$$z = \frac{2}{(1-t)^2}. \tag{14}$$

This is the parabola associated with four lines, which was discussed by Musselman.[3]

If now we choose another point on the cubic (8) given by $t = t_2$ we have associated with this point another parabola

$$z = \frac{2}{(1-t_2)(1-t)^2}. \tag{15}$$

The parabolas (10) and (15) have a common tangent given by

$$z = \frac{2}{(1-t_1)(1-t_2)(1-t)}. \tag{16}$$

It is obvious that any three parabolas associated with three distinct points on the cubic (8) given by t_1, t_2, t_3 will determine three lines each of which will be tangent to two of the parabolas. These three common tangents will meet in a point given by

$$z = \frac{2}{(1-t_1)(1-t_2)(1-t_3)}. \tag{17}$$

Moreover four parabolas determined by t_1, t_2, t_3, t_4 will have six common tangents which intersect by threes in the four points

(18) a) $$z = \frac{2}{(1-t_1)(1-t_2)(1-t_3)}$$

b) $$z = \frac{2}{(1-t_1)(1-t_2)(1-t_4)}$$

c) $$z=\frac{2}{(1-t_2)(1-t_3)(1-t_4)}$$

d) $$z=\frac{2}{(1-t_1)(1-t_3)(1-t_4)}\ .$$

The points (18) all lie on the circle

(19) $$z=\frac{2(1-t)}{P_4}\ ,$$

where $P_4=(1-t_1)(1-t_2)(1-t_3)(1-t_4)$.

The points (18) *a, b, c* determine a triangle, and the Simpson line of (18) *d* with respect to this triangle is

(20) $$zt_2^2-\bar{z}=\frac{2t_2^2-2\sigma_4+\sigma_3-t_2^2\sigma_1}{P_4}\ ,$$ [5]

where $\sigma_1=t_1+t_2+t_3+t_4,\quad \sigma_3=t_1t_2t_3+t_1t_2t_4+t_1t_3t_4+t_2t_3t_4,\quad \sigma_4=t_1t_2t_3t_4.$

The points (18) determine four triangles and in each case the fourth point will determine a Simpson line. These four Simpson lines intersect in the point

(21) $$z=\frac{2-\sigma_1}{P_4}\ .$$

The tangent line to the cubic (8) at t_i is

(22) $$\bar{z}+zt_i^2=\frac{2t_i^2}{(1-t_i)^2}\ .$$

The common tangent to the parabolas at t_i and t_j has the equation

(23) $$zt_it_j+\bar{z}=\frac{2t_it_j}{(1-t_i)(1-t_j)}\ .$$

From Equations (22) and (23) we easily obtain the following Theorem: The tangents to the cubic (8) at the points determined by t_i and t_j and the common tangent to the parabolas at these points form an isosceles triangle.

REFERENCES

1. H. Wieleitner, Spezielle Ebene Kurven, p. 55.
2. G. Loria, Spezielle Algebraische und Transcendente Ebene Kurven, Vol. 1, p. 90.
3. J. R. Musselman, On Four Lines and their Associated Parabola, American Mathematical Monthly, Vol. XLIV, p. 513.
4. F. V. and F. Morley, Inversive Geometry, p. 15
5. J. H. Weaver, Curves Determined by a One-parameter Family of Triangles, American Mathematical Monthly, Vol. XL, p. 85.

A Note on Hummel's Paper

By E. C. Kennedy
Texas College of Arts and Industries

Professor Hummel has shown (*American Mathematical Monthly*, Vol. 47, No. 2) that

$$n! \ > e^{11/12}\sqrt{n}\, n^n e^{-n}, \qquad n \geqslant 2, \tag{1}$$

by making use of the inequality

$$\int_{k-1}^{k} \left[\frac{1}{x} - \frac{2k-1-x}{k(k-1)}\right]^2 dx > 0, \quad k \geqslant 2.$$

In his paper he expresses the belief that his result is new and perhaps the best lower bound of that type so far obtained.

The object of this paper is to get a still better lower bound for $n!$ in the form of (1).

We define a_k, as did Hummel, by the equation

$$\int_{k-1}^{k} \log x \, dx = \tfrac{1}{2}\left[\log(k-1) + \log k\right] + a_k.$$

Thus geometrically, a_k is the area between the curve and the chord.

Let $a_2 + a_3 + \cdots + a_k = A_{2,k}$. It is easy to show that

$$a_k = -1 + (k - \tfrac{1}{2})\log k/k-1$$

and we readily find that

$$A_{2,k} = -k + 1 - \log(k-1)! + (k - \tfrac{1}{2})\log k. \tag{2}$$

Since $A_{2,k}$ is an increasing function of k we now seek a sharp upper bound for $\lim_{k\to\infty} [A_{2,k}] = A_{2,\infty}$. Using Davis': *Tables of Higher Mathematical Functions* and taking $k=100$ we get $A_{2,100} = .080228$. By an elementary process it can be shown that $A_{101,\infty} < .001261$. Hence $A_{2,\infty} < .081489$ and from the fact that

$$\log n! = (n + \tfrac{1}{2})\log n - n + 1 - A_{2,n}$$

we find that

$$\log n! > (n + \tfrac{1}{2})\log(n) - n + .918511. \tag{3}$$

Increasingly accurate rational approximations (from below) are 11/12, 45/49, etc. Thus $e^{45/49}$ would serve better than $e^{11/12}$ as a constant factor in the inequality (1). We obtained the above result without employing Stirling's Formula in any way.

By making use of certain properties of the gamma function, namely,

$$(4)\qquad \log\Gamma(x)=(x-\tfrac{1}{2})\log x-x+\tfrac{1}{2}\log 2\pi+\frac{1}{12x}+\frac{1}{360x^3}\cdots$$

we readily find that

$$(5)\qquad A_{2,k}=1-\tfrac{1}{2}\log 2\pi-\frac{1}{12k}-\frac{1}{360k^3}\cdots\ .$$

Hence $A_{2,\infty}=1-\frac{1}{2}\log 2\pi$, from which we have $1-A_{2,\infty}=.9189385$ and this value is approximated from below by 11/12, 34/37, etc.

Thus we can write

$$(6)\qquad n!>e^{34/37}\sqrt{n}\,n^ne^{-n},\qquad n\geqslant 2,$$

which is a better lower bound than Professor Hummel's.

A Birational T_{n+2} Associated with a Rational Space C_n

By M. L. Vest
West Virginia University

The most general non-involutorial transformation belonging to the complex of lines cutting a twisted cubic curve has been studied by Calderera.* Her transformation of order 15 is determined by two projective pencils of quartic surfaces which contain the cubic C_3 as a double curve, each pair of which contains in addition a line of a regulus of bisecants to the C_3. The same T_{15} is also determined by two pencils of cubic surfaces through the C_3, each pair of which has a common variable double point on C_3.

Since then Davis and Black† have discussed a T_{13} determined by two pencils of quadrics through C_3 and a regulus of bisecants of C_3.

More recently the author‡ of the present paper has studied a T_{10} determined by two pencils of quadrics through C_3, and a T_5 determined by two pencils of planes and the C_3, while Davis and Cunningham§ have discussed a $T_{9,6}$ determined by a C_3, a pencil of planes and a pencil of quadrics.

In the present paper the T_5 mentioned above is generalized by using a rational curve of order n in place of the twisted cubic.

Consider a rational space curve r of order n and two pencils of planes $|\pi|$ and $|\pi'|$, the planes and the points of r being projectively related. Through a generic point $P(y)$ of space passes one π of $|\pi|$. The line t determined by $P(y)$ and the point $P(r)$ of r associated with π cuts the associated plane π' of $|\pi'|$ in a unique point $P'(x)$, the image of $P(y)$ in the T thus defined.

The points of r will in general have unique image points under T. There are, however, $n+1$ points q_i on r whose images under T are planes of $|\pi'|$, i. e., the q_i are fundamental points of the first kind under T. Similarly, r contains $n+1$ points q_i' which are fundamental points of the first kind under T^{-1}.

Let r have as equations

$$(1) \qquad x_1=f_1(u,v),\ x_2=f_2(u,v),\ x_3=f_3(u,v),\ x_4=f_4(u,v),$$

*G. M. Calderera, Circolo Di Matematico Di Palermo, Tomo XVIII, p. 205.

†H. A. Davis and A. H. Black, Allegheny Mountain Section, Math. Assoc. America, October, 1939.

‡M. L. Vest, Allegheny Mountain Section, Math. Assoc. America, April, 1940.

§H. A. Davis and A. B. Cunningham, Allegheny Mountain Section, Math. Assoc. America, April, 1940.

where the f_i are homogeneous functions of order n, and take the pencils of planes as

$$|\pi| \equiv v(ax) - u(bx) = 0 \tag{2}$$

$$|\pi'| \equiv v(a'x) - u(b'x) = 0 \tag{3}$$

where $(ax) \equiv \sum_{i=1}^{4} a_i x_i$ and similarly for (bx), $(a'x)$ and $(b'x)$.

For a point $P(y)$ there is a plane of $|\pi|$ with parameter

$$\frac{u}{v} = \frac{(ay)}{(by)} \tag{4}$$

and to this corresponds a point $P(r)$ on r having as coordinates

$$x_i = z_i, \quad i = 1, \cdots, 4$$

where $z_i \equiv f_i[(ay), (by)]$, and the plane

$$(a'x)(by) - (b'x)(ay) = 0 \tag{5}$$

of $|\pi'|$. The transversal t determined by $P(y)$ and $P(r)$ intersects the plane (5) in the point whose coordinates are

$$T^{-1} : \tau x_i = Qy_i + Kz_i, \qquad i = 1, \cdots, 4. \tag{6}$$

Here $K = (ay)(b'y) - (by)(a'y)$, (order 2) and $Q = (by)(a'z) - (ay)(b'z)$, (order $n+1$) where

$$(a'z) = \sum_{i=1}^{4} a_i' z_i, \text{ etc.}$$

The surface Q consists of $n+1$ planes of $|\pi|$ which are the images of the fundamental points q_i', i. e., $q' \overset{T^{-1}}{\sim} Q$.

Equations (6) are those of the inverse transformation T^{-1} which is seen to be of order $n+2$.

Noting that $K' = -K$ the equations of T, also of order $n+2$, can be written as

$$T : \tau y_i = Q' x_i - K w_i, \qquad i = 1, \cdots, 4 \tag{7}$$

where $w_i \equiv f_i[(a'x)(b'y)]$ and $Q' = (b'x)(aw) - (a'x)(bw)$, (order $n+1$). Q' consists of $n+1$ planes of $|\pi'|$ and $q \overset{T}{\sim} Q'$.

If we apply the transformation T^{-1} to a plane $(a'x)$ of $|\pi'|$ we obtain

$$(a'x) \overset{T^{-1}}{\sim} (ay)L$$

where $L=(b'y)(a'z)-(a'y)(b'z)$, a surface of order $n+1$. Here (ay) is the corresponding plane of $|\pi|$, hence

$$1' \overset{T^{-1}}{\sim} L$$

where $1'$ is the base of $|\pi'|$. Similarly, $(ay) \overset{T}{\sim} (a'x)L'$ and $1 \overset{T}{\sim} L'$.

The pointwise invariant surface of the transformation is the quadric $K=0$. We find that

$$K' \overset{T^{-1}}{\sim} KLG$$

where $G=(by)(az)-(ay)(bz)$, a surface consisting of $n+1$ planes of $|\pi|$ which are the images under T^{-1} of $n+1$ lines g_i' and contain the fundamental points q_i. The lines g' are the intersections of corresponding planes of Q' and G. That is

$$g' \overset{T^{-1}}{\sim} G.$$

Similarly $K \overset{T}{\sim} KL'G'$ and $g \overset{T}{\sim} G'$, G' consisting of $n+1$ planes of $|\pi'|$ passing through q', the planes being the images of the lines g which are the intersections of corresponding planes of Q and G'.

The above statements may be expressed in the form

$$[Q_i, G_i'] : g_i, \qquad i=1,\cdots,n+1.$$

$$[Q_i', G_i] : g_i', \qquad i=1,\cdots,n+1.$$

Furthermore

$$G' \overset{T^{-1}}{\sim} L^{n+1}Q, \quad G \overset{T}{\sim} L'^{n+1}Q'$$

and also

$$L' \overset{T^{-1}}{\sim} QL^nG, \quad L \overset{T}{\sim} Q'L'^nG'.$$

A generic plane $\pi' \equiv (Ax)=0$ under T^{-1} gives

$$\pi' \overset{T^{-1}}{\sim} Q(Ay)+K(Az)=\phi_{n+2}$$

the homaloidal web of T^{-1}. Further

$$\phi \overset{T}{\sim} Q'L'^{n+1}G'(Ax).$$

Hence the homaloidal web is

$$\infty^3|\phi| : 1^{n+1}g.$$

The intersection of two surfaces of the web gives

$$H \equiv [\phi,\phi] : 1^{n^2+2n+1}gc_{n+2}.$$

The jacobian of T consists of $Q'^2L'G'$.

The correspondences can now be summarized as follows:

$$1 \overset{T}{\sim} L' : 1'^n 1 g' \qquad K \overset{T}{\sim} K : 1'1g'g$$

$$\pi \overset{T}{\sim} \phi' : 1'^{n+1} g' \qquad g_i \overset{T}{\sim} G_i' : 1'g_i$$

$$q_i \overset{T}{\sim} Q' : 1'g_i'.$$

The intersection table follows:

$$[L', K] : 1'^n 1 g' \qquad [K, G_i'] : 1'g_i$$

$$[L', \phi'] : 1'^{n^2+n} g' c_{n+1} \qquad [K, Q_i'] : 1'g_i'$$

$$[1', G_i'] : 1'^n k_1 \qquad [\phi', Q_i'] : 1'^{n+1} g_i'$$

$$[L', Q_i'] : 1'^n g_i' \qquad [\phi', G_i'] : 1'^{n+1} c_1$$

$$[K, \phi'] : 1'^{n+1} g' k_2 \qquad [G_i', Q_i'] : 1'.$$

Humanism and History of Mathematics

Edited by
G Waldo Dunnington

A History of American Mathematical Journals

By Benjamin F. Finkel
Drury College

(Continued from April, 1942, issue)

THE

MATHEMATICAL MAGAZINE

A JOURNAL OF

ELEMENTARY MATHEMATICS

Edited and Published

by

Artemas Martin, M. A., Ph.D., LL.D.,

Member of The London Mathematical Society, Member of The Edinburg Mathematical Society, Member of the Mathematical Society of France, Member of The Philosophical Society of Washington

VOLUME I

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Washington, D. C.

1887

The *Mathematical Magazine* was founded by Artemas Martin, M. A., Ph.D., LL.D., Washington, D. C., in 1887, though the first number bears the date, January, 1882, and was edited and published by him during the entire time of its existence. Apparently, while he was publishing the *Magazine* he was still continuing the publication of the *Mathematical Visitor*.

(After the review of Volume I of the *Mathematical Visitor* was printed, the Writer, through the courtesy of Dr. Walter F. Shenton, of the American University, Washington, D. C., received the four Numbers of Volume II of the *Mathematical Visitor*. No. 1, Vol. II, bears the date, January, 1882; No. 2, January, 1883; No. 3, January, 1887; and No. 4, January, 1894.

These four Numbers contain a total of 122 pages and very probably are all of Vol. II that were ever published.

In addition to interesting Editorial Items and brief reviews of Books and Periodicals, these Numbers like the whole of Volume I, are replete with very interesting problems many of them accompanied with most excellent solutions.

The Editor informs his readers that the typesetting of No. 1, Vol. I, of the *Magazine* was done with his own hands and the printing in the office of the *Erie Advertiser*, although executed by "professionals", it was not as good as it might be. He felt frequently obliged to apologize to his Readers for the irregularity with which it reached them because of his illness and other unavoidable circumstances. Yet when one recalls that probably most, if not all, of the typesetting and very much of the press-work of both *The Mathematical Visitor* and the *Mathematical Magazine* were done with the Editor's own hands, one marvels at his accomplishments and the excellence of the character of his work.

The Writer of this *History* is proud to say that he was a contributor, though a very feeble one, of Problems and Solutions to Dr. Martin's Publications. But during the early years of his forty-two years of service to Drury College, being obliged to carry from 18 to 27 hours of class-room work per week, in the College, acting as Secretary of the Faculty, Registrar of the College, Nominal Librarian for a number of years, doing his share of committee work, Director of Summer Sessions, 1916-1918, and editing and publishing the *American Mathematical Monthly*, he was limited by the Law of the Conservation of Energy of doing more.) F.

It was the purpose of the Editor to issue the *Mathematical Magazine* quarterly but on page 67, No. 4, Vol. I, he said: "As the 4 Nos.

of one year will make too thin a volume for binding, it is proposed to make 12 Nos. constitute a volume, and number and page them consecutively. The next No., therefore, will be No. 5 of Vol. 1.

Following the Title Page, pp. (iii), (iv), (v) is a list of 144 Contributors, to Volume I; then p. vii, Contents of Volume I., in which is listed the Introduction by the Editor.

Since the Editor states his aim and purpose for founding the *Magazine* in his Introduction, we will here set forth his Introduction which reads as follows:

"Periodical Publications have done much in other countries to stimulate a desire for mathematical learning, and some of our own best mathematicians commenced a distinguished career by solving the problems appearing in similar works.

The most noted of these issued in England were the *Lady's Diary*, the *Gentlemen's Diary*, the *Mathematical Companion*, the *Mathematical Repository*, the *Mathematician*,—all which have long since been discontinued and are now out of print.

In this country we have had the *Mathematical Correspondent*, the *Mathematical Diary*, the *Mathematical Miscellany*, the *Mathematical Monthly*, etc.; and *now* have the *Analyst*, the *American Journal of Mathematics*, and *The Mathematical Visitor*.

An elementary mathematical periodical is a desideratum. No such periodical is at present published in the English language. The existing mathematical journals are too far above the average teacher and student.

In order to supply, in a measure, this long-felt want, the *Mathematical Magazine* is offered to the mathematical public. It will be issued in quarterly numbers of 16 or 20 quarto pages (pages 9″x12″), at $1.00 a year, payable in advance, and devoted to the elementary branches of mathematics, viz.: Arithmetic, Algebra, Geometry, Trigonometry, etc.; and will contain Problems and Solutions, Notes, and Papers, from some of the best writers in this and other countries.

One of the features of the *Mathematical Magazine* will be the solution and discussion of such of the problens found in the various text-books in use as are of special interest, or present some difficulty.

The Editor will endeavor, to the best of his ability, to make the *Magazine* interesting and instructive, but in order to be successful he must have the earnest co-operation of his readers. Teachers, students, and all lovers of Mathematical Science are, therefore, hereby invited to contribute problems, solutions, and articles on interesting and important subjects of an elementary character.

A wide range of subjects will have to be considered, and no reader should be discouraged or dissatisfied if he meets with an article that is not in his particular line of research. No *one* person should expect the space of such a periodical as this to be devoted exclusively to the subjects that are especially interesting to himself individually.

Occasionally an article on the elements of some of the higher branches of Mathematics will be given for the purpose of pointing the student 'onward and upward'."

ARTEMAS MARTIN.

List of Contributors, (iii-iv).

The list of contributors consists of 144 mathematicians, mostly American. We shall give the names of a score or more of the most prominent as follows: Professor Edward E. Bowser, Professor of Mathematics, Rutgers College, New Brunswick, N. J.; Dr. Edward Brooks, later superintendent of schools, Philadelphia, Penn.; Professor W. W. Beman, Assistant Professor, later Professor of Mathematics, University of Michigan; Marcus Baker, U. S. Coast and Geodetic Survey Office, Washington, D. C.; Dr. Joseph Ficklin, Professor of Mathematics and Astronomy, Missouri State University, Columbia, Missouri; Dr. Joel E. Hendricks, Editor and Publisher of the *Analyst*, Des Moines, Iowa; Henry Heaton, Perry, Iowa; Professor William Hoover, later, Professor of Mathematics, Ohio University, Athens, Ohio; Dr. David S. Hart, Stonington, Conn.; Professor William Woolsey Johnson, Professor of Mathematics, St. John's College, Annapolis, Md.; Dr. Daniel Kirkwood, Professor of Mathematics, University of Indiana, Bloomington, Ind.; Charles H. Kummell, U. S. Coast and Geodetic Survey Office, Washington, D. C.; Miss Christine Ladd, Fellow of Johns Hopkins University, later Professor of Natural Science and Chemistry, Howland School, Union Springs, N. Y.; Dr. Alexander Macfarlane, Edinburg, Scotland; F. P. Matz, Professor of Mathematics, Military and Scientific School, King's Mountain, N. C.; Professor Benjamin Peirce, Professor of Mathematics, Harvard University; J. J. Sylvester, LL.D., Professor of Mathematics, Johns Hopkins University; E. B. Seitz, Professor of Mathematics, North Missouri State Normal School, Kirksville, Missouri; Professor Ormond Stone, Astronomer at the Cincinnati Observatory; Walter Siverly, Oil City, Penn.; Professor De Volson Wood, Professor of Mathematics and Mechanics, Stephens Institute of Technology, Hoboken, N. J.; Dr. S. H. Wright, late Mathematical Editor of the *Yates County Chronicle*, Penn Yan, N. Y.

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The problem is stated as follows: Suppose one cent to have been placed at 6% compound interest, at the commencement of the Christian era. What would the amount have been January 1, 1882?

Professor Seitz computed the amount and found it to be $\$398413 \times 10^{39}$ which he called the result in "round numbers". In order to assist the imagination, he assumed the value of a cubic foot of gold to be \$360000. He then assumed the earth to be of solid gold and computed its value on that basis and found that that enormous amount, "when compared with the amount of our one cent, is but a *drop in the ocean*". He then inquires how many golden suns would be required to pay the amount.

He gives as a concluding illustration, the radius of a sphere of gold whose value would be equal to the amount of the cent.

He then draws the student's attention to the fact that if this imaginary sphere were placed with its center at the center of the sun, its surface would extend far beyond the orbit of the planet Uranus, taking in all the planets of the solar system, except Neptune.

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18 Problems Proposed for Solutions, pp. 256-257; About Sixth-Power Numbers Whose Sum is a Sixth Power, by Artemas Martin, pp. 265-271; Elementary Proof of Two Properties of Numbers, by Rollin A. Harris, Ph.D., U. S. Coast and Geodetic Survey Office, Washington, D. C., p. 272; The United States Sinking-Fund, by Theodore L. DeLand, Office of the Secretary of the Treasury, Washington, D. C., pp. 273-274; Rational Scalene Triangles, by Artemas Martin, pp. 275-284; On Biquadrate Numbers, by Cyrus Benton Haldeman, Ross, Ohio, pp. 285-296; On Rational Right-Angled Triangles, by Artemas Martin, pp. 297-300; Table of Prime Rational Right-Angled Triangles, pp. 301-308; Properties of Right-Angled Triangles, pp. 309-321; Table of Rational Right-Angled Triangles Whose Legs are Consecutive Whole Numbers, pp. 322-324; About Biquadrate Numbers Whose Sum is a Biquadrate.—II, by Artemas Martin, pp.325-352.

The Writer after having received No. 12 of Vol. II of the *Magazine* bearing the date January, 1904, and apparently completed and consisting of pp. 265-352 had the volume bound. Afterwards he received Algebraic Solution of the Celebrated "Three Point" Problem, by Artemas Martin, bearing the date of 1911 and the paging being the continuation of the the paging of No. 12, Vol. II, pp. 353-360. On the front cover it is stated (From the *Mathematical Magazine*, Vol. II, No. 12). Under the title of the article it is stated (Contributed to Section A of the American Association for the Advancement of Science at the Meeting held at Baltimore, Md., December 28, 1908 to January 2, 1909; also, to the Edinburg Mathematical Society, February 10, 1911. An interesting bibliography of the problem precedes the solution.

Dr. Shenton in a letter to the Writer, March 2, 1942, says that Dr. Martin continued to publish additions to the *Magazine* thus increasing the number of pages to Vol. II through 396. Being a typesetter and a press-man himself, he could print additions whenever the spirit moved him. F.

The Highest Rung

By WILLIAM L. SCHAAF
Brooklyn College, Brooklyn, N. Y.

The nearer man approaches mathematics the farther away he moves from the animals—S. Casson, *Progress and Catastrophe.*

"The mathematician has reached the highest rung on the ladder of human thought. But it is the same ladder on which all of us have been always ascending, alike from the infancy of the individual and the infancy of the race. Molier's *Jourdain* had been speaking prose for forty years without knowing it. Mankind has been thinking poetry throughout its long career and remained equally ignorant." So writes Havelock Ellis in his *Dance of Life,* alluding to the abstract nature of mathematics and to the austerity of rigorous thinking.

What, exactly, is the intrinsic nature of mathematics? What, precisely, are its intellectual and cultural imports? The answers are not as readily given as might be supposed. Somewhere Keyser has said that mathematics could be regarded either as an enterprise or as a body of achievements. "As an enterprise mathematics is characterized by its aim, and its aim is to think rigorously whatever is rigorously thinkable or whatever may become rigorously thinkable in course of the upward striving and refining evolution of ideas. As a body of achievements mathematics consists of all the results that have come, in the course of the centuries, from the prosecution of that enterprise: the truth discovered by it; the doctrines created by it; the influence of these, through their applications and their beauty, upon the advancement of civilization and the weal of man."

The domain of mathematical achievement, as it stands today, is unbelievably vast and extensive. It embraces ramifications that stagger the imagination and beggar description. No one, even within the span of a lifetime, could become genuinely familiar with all of its major branches, let alone master them. The day of the universalist, of a Gauss or a Poincaré, is indubitably past. As Sarton has aptly put it: "The mathematical universe is already so large and diversified that it is hardly possible for a single mind to grasp it, or, to put it in another way, so much energy would be needed for grasping it that there would be none left for creative research. A mathematical congress of today reminds one of the Tower of Babel, for few men can follow profit-

ably the discussions of sections other than their own, and even there they are sometimes made to feel like strangers." And strangely enough, most of this colossal achievement, when reckoned by the Ages, was born of the morning. For E. T. Bell reminds us that "there are good grounds for the frequent assertion that the nineteenth century alone contributed about five times as much to mathematics as had all preceding history. This applies not only to quantity, but, what is of incomparably greater importance, to power." Sixty years ago Arthur Cayley, in an inaugural address before the British Association (1883), enthusiastically declared that "it is difficult to give an idea of the vast extent of modern mathematics. This word 'extent' is not the right one: I mean extent crowded with beautiful detail,—not an extent of mere uniformity such as an objectless plain, but of a tract of beautiful country seen at first in the distance, but which will bear to be rambled through and studied in every detail of hillside and valley, stream, rock, wood, and flower." Needless to say, undreamt of vistas have been opened and explored since Cayley's time.

All attempts to define mathematics, in the customary sense of a definition, must apparently be abandoned. Picturesque descriptions have been proferred from time to time: Benjamin Peirce's "mathematics is the science which draws necessary conclusions"; W. B. Smith's "Mathematics is the universal art apodictic"; or Bertrand Russel's "mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true." Suggestive, also, is the unwitting aphorism of Peirce, when he remarked to his students, having demonstrated the relation $e^{\pi/2} = \sqrt[i]{i}$: "Gentlemen, that is surely true, it is absolutely paradoxical, we cannot understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth". Nevertheless, our task is not altogether hopeless. Instead of defining mathematics, we shall try to point out some of those qualities which characterize it uniquely.

Prominent among such qualities we would put its *fictional nature;* mathematics is a deliberate, arbitrary creation of the human mind. In a very real sense completely detached from persons or things, it has an essentially independent existence apart from experience, from which detachment arise an indigenous self-sufficiency and an intrinsic self-consistency that are unique among all human achievements. This aloofness—this impersonal quality—is aptly suggested by the following cogent passage:* "Mathematical truth has validity independent of place, personality, or human authority. Mathematical

**Mathematics in General Education.* Report of the Progressive Education Association. D. Appleton-Century Co., 1940, p. 256. By permission.

relations are not established, nor can they be abrogated, by edict. The multiplication table is international and permanent, not a matter of convention nor of relying upon authority of state or church. The value of π is not amenable to human caprice. The finding of a mathematical theorem may have been a highly romantic episode in the personal life of the discoverer, but it cannot be expected of itself to reveal the race, sex or temperament of this discoverer."

A further characteristic of mathematics may be said to be its *matchless succinctness;* both as to symbols and to words, mathematics is unrivaled in this respect by any other human creation. The pregnant fertility of mathematical symbols, laden with meaning and fraught with implications, can be felt even by the tyro; the compactness and sheer force of an "elegant" proof intrigues the more serious student, to whom the experience is well known. The very simplicity of the words frequently used to represent highly abstract and subtle concepts is utterly astounding. Professor Kasner puts it quaintly: "mathematics is the science which uses easy words for hard ideas." The uninitiated could not possibly suspect the wealth of connotation assigned to common everyday words such as family, range, region, series, space, field, group, set, domain, ring, class, order sequence, net, ray, bundle, pencil, dense, cut, mild, compact. Consider, for example, the term *limit;* one of the several mathematical meanings of this simple word is as follows: If V be a variable whose range R is included in the field F of a sequence S; and if an F term t be such that, given any S predecessor t' of t among the R terms, there is an R term between t' and t, or such that, given any S successor t' of t among the R terms, there is an R term between t and t', then t is an S limit of V. Or again take the homely word *tube,* which to the unsuspecting layman would suggest variously a pipe, an inner tire, or a tunnel under a river; ask the mathematician and he will tell you that "with reference to a curve C with continuously turning tangent in a metrical space of any number of dimensions, we define a tube as the locus of points at a fixed distance Θ, called the radius, from C, the distance being measured in each case along a geodesic perpendicular to C."

Most significant perhaps of all the aspects of mathematics, however, are its *poetic nature,* its sheer *imaginative qualities,* which soar to limitless heights of beauty. This capacity to transcend earthly realities and human limitations, to pass into lofty spiritual realms, cannot readily be described or conveyed. We must content ourselves to let the masters and their disciples speak for themselves, staccato-like, in their own inimitable way.

. . . .

The mathematician's best work is art, a high and perfect art, as daring as the most secret dreams of imgagination, clear, and limpid. Mathematical genius and artistic genius touch each other.—MITTAG-LEFFLER.

Does it not seem as if Algebra had attained to the dignity of a fine art, in which the workman has a free hand to develop his conceptions, as in a musical theme or a subject for a painting? It has reached a point where every properly developed algebraical composition, like a skillful landscape, is expected to suggest the notion of an infinite distance lying beyond the limits of the canvas.—J. J. SYLVESTER.

.... the simplicity, the indispensableness of each word, each letter, each little dash, that among all artists raises the mathematician nearest to the World-creator; it establishes a sublimity which is equalled in no other art,—Something like it exists at most in symphonic music.—BOLTZMAN

If we compare a mathematical problem with an immense rock, whose interior we wish to penetrate, then the work of the Greek mathematicians appears to us like that of a robust stonecutter, who, with indefatigable perseverance, attempts to demolish the rock gradually from the outside by means of hammer and chisel; but the modern mathematician resembles an expert miner, who first constructs a few passages through the rock and then explodes it with a single blast, bringing to light its inner treasures.—HANKEL.

Poesy has been called a creation, a making, a fiction; and the Mathematics have been called .... the sublimest and the most stupendous of fictions.—THOMAS HILL.

Music is the pleasure the human soul experiences from counting without being aware that it is counting.—LEIBNITZ.

Mathematics, too, has its triumphs of the creative imagination, its beautiful theorems, its proofs and processes whose perfection of form has made them classic. He must be a "practical" man who can see no poetry in mathematics.—W. F. WHITE.

There is an astonishing imagination, even in the science of mathematics .... We repeat, there was far more imagination in the head of Archimedes than in that of Homer.—VOLTAIRE.

Dans les Mathématiques, la censure et la critique ne peuvent être permises a tout le monde; les discours des rhéteurs et les défenses des avocats n'y valent rien.—FRANCOIS VIETÉ.

A mathematician who is not also something of a poet will never be a complete mathematician.—KARL WEIERSTRASS.

Let us grant that the pursuit of mathematics is a divine madness of the human spirit, a refuge from the goading urgency of contingent happenings.—A. N. WHITEHEAD.

The most distinct and beautiful statement of any truth must take at last the mathematical form.—THOREAU.

Architecture is geometry made visible in the same sense that music is number made audible.—CLAUDE BRAGDON.

The domain of mathematics is the sole domain of certainty. There and there alone prevail the standards by which every hypothesis respecting the external universe and all observation and all experiment must be finally judged. It is the realm to which

all speculation and thought must repair for chastening and sanitation, the court of last resort, I say it reverently, for all intellection whatsoever, whether of demon, or man, or deity. It is there that mind as mind attains its highest estate.—C. J. KEYSER.

The mathematician is fascinated with the marvelous beauty of the forms he constructs, and in their beauty he finds everlasting truth.—J. B. SHAW.

In the pure mathematics we contemplate absolute truths, which existed in the Divine Mind before the morning stars sang together, and which will continue to exist there, when the last of their radiant host shall have fallen from Heaven.—E. T. BELL.

Art is an expression of the world order and is, therefore, orderly, organic, subject to mathematical analysis.—CLAUDE BRAGDON.

In the sphere of mathematics we are among processes which seem to some the most inhuman of all human activities and the most remote from poetry. Yet it is just here that the artist has the fullest scope for his imagination. We are in the imaginative sphere of art, and the mathematician is engaged in a work of creation which resembles music in its orderliness. It is not surprising that the greatest mathematicians have again and again appealed to the arts in order to find some analogy to their own work. They have indeed found it in the most varied arts, in poetry, in painting, and in sculpture, although it would certainly seem that it is in music, the most abstract of all the arts, the art of number and time, that we find the closest analogy.—HAVELOCK ELLIS.

Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show .... Remote from human passions, remote even from the pitiful facts of nature, the generations have gradually created an ordered cosmos, where pure thought can dwell as in its natural home, and where one, at least, of our nobler impulses can escape from the dreary exile of the natural world.—BERTRAND RUSSELL.

Conterminous with space and coeval with time is the kingdom of Mathematics; within this range her dominion is supreme; otherwise than according to her order nothing can exist; in contradiction to her laws nothing takes place. On her mysterious scroll is to be found written for those who can read it that which has been, that which is, and that which is to come.—W. SPOTTISWOODE.

Mathematics represents one of the great imaginative activities of mankind. It is the language of ideas. In ordinary language words are symbols which stand for abstractions of varying degree. In mathematics, symbols are chosen which represent ideas abstractly also, but more simply because they carry in themselves and their combinations the logical connection of ideas. In difficult processes of thought no other tool is adequate.—G. C. EVANS.

In mathematics it is notorious that we start from absurdities to reach a realm of law, and our whole (mathematical) conception of the world is based on a foundation which we believe to have no existence .... Fiction is, indeed, an indispensable supplement to logic, or even a part of it; whether we are working inductively or deductively, both ways hang closely together with fiction; and axioms, though they seek to be primary verities, are more akin to fiction. If we had realized the nature of axioms, the doctrine of Einstein, which sweeps away axioms so familiar to us that they seem obvious

truths, and substitutes others which seem absurd because they are unfamiliar, might not have been so bewildering.—HAVELOCK ELLIS.

Each thing in the world has names or unnamed relations to everything else. Relations are infinite in number and kind. To be is to be related. It is evident that the understanding of relations is a major concern of all men and women. Are relations a concern of mathematics? They are so much its concern that mathematics is sometimes defined to be the science of relations.—C. J. KEYSER, *Mole Philosophy*.

Given any domain of thought in which the fundamental objective is a knowledge that transcends mere induction or mere empiricism, it seems quite inevitable that its processes should be made to conform closely to the pattern of a system free of ambiguous terms, symbols, operations, deductions; a system whose implications and assumptions are unique and consistent; a system whose logic confounds not the necessary with the sufficient where these are distinct; a system whose materials are abstract elements interpretable as reality or unreality in any forms whatsoever provided only that these forms mirror a thought that is pure. To such a system is universally given the name MATHEMATICS.—S. T. SANDERS.

The problems of the infinite have challenged man's mind and have fired his imagination as no other single problem in the history of thought. The infinite appears both strange and familiar, at times beyond our grasp, at times easy and natural to understand. In conquering it, man broke the fetters that bound him to earth. All his faculties were required for this conquest—his reasoning powers, his poetic fancy, his desire to know.—KASNER and NEWMAN, *Mathematics and the Imagination*.

This relation logical implication is probably the most rigorous and powerful of all the intellectual enterprises of man. From a properly selected set of the vast number of propositional functions a set can be selected from which an infinitude of propositional functions can be implied. In this sense all postulational thinking is mathematics. It can be shown that doctrines in the sciences, natural and social, in history, in jurisprudence and in ethics are constructed on the postulational thinking scheme and to that extent are mathematical. Together (the proper enterprise of science which is largely empirical and the enterprise of mathematics which is categorical) they embrace the whole knowledge-seeking activity of mankind, whereby knowledge is meant the kind of knowledge that admits of being made articulate in the form of propositions.—C. J. KEYSER.

The Teacher's Department

Edited by
JOSEPH SEIDLIN, JAMES MCGIFFERT
J. S. GEORGES and L. J. ADAMS

Adequate Induction by Sampling

By ISIDOR F. SHAPIRO
New York City

Scope: "Induction" as here used refers to "proof by induction," that is to say, *tests* showing whether a proposed serial equation is conditional, or whether it is true unconditionally for all values of n.

I. Intuitive Induction is Ordinarily not Reliable: When students are introduced to the method of proof by Mathematical Induction, they can be awakened to the need of its rigor if they be first given an exercise or two like testing the following *false* propositions. These have the pitfall of being apparently verified by intuitive sampling, when we make the natural trials of substituting $n=1,2,3,4$; but they collapse when $n=5$.

$$(1)\qquad 6+12+24+42+\cdots+3(n^2-n+2)=\frac{n^4+35n^2+24}{10}.$$

$$(2)\qquad 4+2+10+14+\cdots+[2^n-2(-1)^n]=\frac{42-52n+24n^2-2n^3}{3}.$$

$$(3)\qquad 5+7+17+31+\cdots+[2^{n+1}+(-1)^{n+1}=\frac{n}{6}(46-29n+14n^2-n^3).$$

II. Conditions for Adequacy of Intuitive Induction. Sometimes the inordinate length of a proof by Mathematical Induction makes a shorter proof desirable; and we can take advantage of the fact that there are certain circumstances where induction by sampling-tests is authentic. These conditions obtain when the number of the samples verified is merely one more than the number, k, of generalized parameters in the equation, after all terms have been brought to one side.

For example (though in this case the standard method of proof is not laborious), let us take the serial proposition,

$$1+2+3+4+\cdots+n=\frac{n^2+n}{2}\ .$$

This corresponds to a generalized equation, $p_1n^2+p_2n+p_3=0$, yielding at most $k=3$ parameters. Accordingly one more, viz., 4 samplings, e. g., $n=1,2,3,4$, suffice to check and prove the proposition.

In contrast, Ex. (3) above can be parameterized as

$$2^{n+2}+p_1(-1)^{n+2}+p_2+p_3n+p_4n^2+p_5n^3=0,$$

thus yielding 5 mutable parameters. Then Ex. (3) would indicate the need of 6 samples holding true. For practical purposes we would start our trials with $n=6$, and thus immediately disprove it, avoiding the need of further trials.

We are thus enabled to determine the value of $(k+1)$, that is, the number of sampling-tests that will be "sufficient" to prove a serial proposition. But it is "necessary" that precaution be taken that, on the serial side of the proposition, the highest exponent occurring in the nth term is less than or equal to the highest exponent in the generalized parametric expression presented for determining k.

The proof of the validity of this method of induction follows from the principles of curve-fitting and extrapolation, and depends primarily upon the last-mentioned "necessary" condition.

It should be noted that the method here presented is distinct from the standard Mathematical Induction by recurrence. But it is a mathematically sound and time-saving method, worth distinguishing as "Adequate Induction".

The Construction and Use of a Mathematics Placement Test

By Brewster H. Gere
Herzl Junior College, Chicago, Illinois

There are two mathematics courses, not including special courses such as Business Mathematics, which may be taken by entering students at Herzl Junior Colllege. Normally, a student with two semesters of Algebra and two semesters of Plane Geometry in high school should take a beginning three hour course called Mathematics 101. A student with three semesters of Algebra and two semesters of Plane Geometry usually should take a more advanced course called Mathematics 111. The completion of the 101 course is an equivalent prerequisite for entrance into 111. This 111 course is unified freshman mathematics comparable in difficulty with College Algebra or Trigonometry.

To determine before registration which of these courses should be taken by a student whose preparation was irregular, a placement test was devised. The results were good enough to justify the use of a placement test for all students, regular or irregular in preparation, who had not been enrolled previously in our mathematics courses. The purpose of this paper is to give the method of construction of the test and a discussion of some of the more important results.

In the first draft, a separate test was given for each of the two courses. Both tests were elementary, with few word problems and with questions involving mathematical concepts limited in scope. These forms were designed to eliminate students with flagrant inadequacies of preparation. Certain improvements seemed to be necessary. The most important were to combine the two tests for ease of administration, to broaden the scope, to remove unreliable questions, and to obtain a sharper diagnostic property.

A trial draft of the present form was given as a final examination in the 101 course in January, 1941. Some questions were similar to the best of those in the two original tests. Many additional questions were obtained from various sources, including faculty members in other departments. By discarding some questions and rewriting others, the present form was derived directly from this trial form. The easier questions were placed in the first part of the test whenever logical

grouping was not disturbed. The procedure used in determining the difficulty of a question was merely to count the number of wrong answers and omissions for that question. The test can be answered in an hour by almost all of the students and it has been found easy to administer and to score.

In determining whether or not a given question should be retained in the test, a simple but effective index was calculated. First the papers were ranked on a basis of total score. The highest quarter and lowest quarter only were retained for further analysis. Then for each question those papers in the highest quarter with the correct answer were counted. Call this R_u. For the same question those papers in the lowest quarter without the correct answer were counted. Call this number W_L. The index I is then given by the formula $I = R_u + W_L$. The higher the index, the more reliable the question is assumed to be.

The final form of the test consists of sixty questions, scored either right or wrong, of which the following are examples:

Question 5. $3^9/3^7 = \ldots\ldots$

Question 7. Find the value of a if $3a = 8 - a$.

Question 23. Factor $15y^2 - 2xy - x^2$.

Question 30. Is the following true or false?

$$\frac{2x-y}{4} = \frac{x-y}{2}.$$

Question 42. If an automobile uses 8 gallons of gasoline to travel 144 miles, how far can it travel in using 10 gallons?

Other representative items are given at the end of this paper.

The test was given to 57 entering students during the registration period in February, 1941. The difficulty of each question was then calculated. This group includes both those who intended to take 101 and those who intended to take 111. Other students took the test immediately after classes had begun a week later. Separate indices of reliability were calculated for the 101 and the 111 students. These results are shown in Table I.

It can be seen that some questions which are good for 111 are not very satisfactory for 101 and vice-versa. This might have been expected. If only one group were tested, or if one desired to score a particular group using less than the sixty questions, it would be a simple matter to disregard in scoring, say, ten questions with low indices for such a group.

To show the success of the test in forecasting future performance, in Table II a comparison of grades in the test with grades obtained at the end of the semester is given for 91 students who took Mathematics

TABLE I

Question Number	*Diffi-culty*	*Index 111*	*Index 101*	*Question Number*	*Diffi-culty*	*Index 111*	*Index 101*
1	10	32	31	31	11	32	48
2	14	44	34	32	27	46	39
3	15	36	44	33	34	50	37
4	17	44	48	34	18	34	45
5	22	46	40	35	11	42	44
6	13	44	48	36	22	46	37
7	9	26	47	37	12	38	45
8	8	34	52	38	21	44	38
9	16	44	40	39	13	40	47
10	22	42	46	40	30	50	36
11	23	46	42	41	29	48	43
12	19	42	40	42	3	32	35
13	20	44	45	43	17	44	40
14	30	50	35	44	10	38	39
15	3	52	35	45	30	48	46
16	49	42	32	46	37	42	25
17	19	44	48	47	55	32	28
18	20	50	44	48	46	42	30
19	24	42	43	49	49	36	29
20	27	54	40	50	28	48	37
21	21	42	47	51	23	46	38
22	25	48	46	52	19	44	36
23	31	42	41	53	45	34	31
24	24	46	48	54	34	46	34
25	28	48	38	55	38	42	31
26	37	38	35	56	44	50	28
27	12	34	47	57	34	54	33
28	21	44	38	58	34	42	32
29	15	34	38	59	49	40	30
30	22	42	47	60	45	42	30

Difficulty = Number wrong of 57 papers (both 111 and 101).
Index = Right in highest quarter + Wrong in lowest quarter.
Maximum for each index is 56.
Basis for 111 index: 56 papers.
Basis for 101 index: 112 papers.

101. Since course grades are by letter, the test scores are translated to letter grades in the table.

It is to be noted how few students moved more than one letter classification from beginning to end. Also, the high proportion of drops in the D and F groups indicate that poor preparation is a powerful reason for a lack of interest.

TABLE II

Placement test grade	*Course grade (101)*						*Number of students*
	A	*B*	*C*	*D*	*F*	*F (due to absence)*	
35-60 A	5	..	1	..	..	..	6
24-34 B	1	7	10	..	..	1	19
14-23 C	..	10	18	9	2	1	40
7-13 D	..	..	7	8	1	4	20
0- 6 F	..	..	..	3	..	3	6
Number of students	6	17	36	20	3	9	91

To analyze the results of the test from another point of view, the mathematical concepts and fields of information necessary to answer the question items were listed. The question numbers were also listed in order of difficulty and for each question check marks were placed

TABLE III

Concepts	*Occurrence of concepts in question groups* (Groups of fifteen questions ranked easy to difficult)				
	I	*II*	*III*	*IV*	*Total*
Arithmetic	4 items	2 items	1 items	0 items	7 items
Geometry	3 "	2 "	2 "	0 "	7 "
Knowledge of formulas	0 "	2 "	2 "	0 "	4 "
Literal substitution	3 "	1 "	3 "	2 "	9 "
Algebraic operations	4 "	5 "	3 "	1 "	13 "
Fractions (numerical)	1 "	2 "	1 "	0 "	4 "
Fractions (literal)	0 "	0 "	1 "	2 "	3 "
Linear equations	3 "	0 "	2 "	2 "	7 "
Simultaneous linear equations	0 "	0 "	2 "	0 "	2 "
Quadratic equations	0 "	0 "	0 "	4 "	4 "
Progressions	0 "	1 "	0 "	0 "	1 "
Table of values	0 "	1 "	0 "	0 "	1 "
Coordinate system	0 "	0 "	0 "	1 "	1 "
Variation and proportion	2 "	1 "	2 "	3 "	8 "
Word problems	2 "	0 "	4 "	3 "	9 "
Exponents and logarithms	0 "	0 "	0 "	3 "	3 "

opposite each concept needed to answer that particular item. An abbreviation of this table is shown in Table III, in which the questions of the test are divided into quarters. Group I consists of the fifteen least difficult questions, whereas Group IV is the most difficult group of fifteen. The number of checks in each group are recorded for the various concepts. For example, the knowledge of the processes of arithmetic are needed to answer correctly four questions in the least difficult group of fifteen questions, two questions in the next group of fifteen questions, one question in the third group, and no questions in the group of fifteen which are the most difficult. This does not mean that arithmetic computation is not involved in any of the fifteen most difficult questions, but that arithmetic is not one of the essential features of these questions.

By assuming that a concept is well understood by a large percentage of the students taking the test if it occurs for the most part in the easier groups, but is not well understood if it occurs frequently in the more difficult groups, a few tentative conclusions may be drawn.

1. Ordinary arithmetic computation is reasonably good.
2. Simple geometrical ideas, such as formulas for area, the meaning of similarity, and the sum of the angles of a triangle, are known.
3. The concept of using letters to represent numbers is not thoroughly understood.
4. The mechanical operations of algebra are usually well learned.
5. Fractions with numerical denominators are of moderate difficulty, but the introduction of literal expressions in the denominators gives great difficulty.
6. Quadratic equations are not solved properly.
7. The Cartesian coordinate system and graphs of equations are rarely understood.
8. The idea of variation and proportion is probably not well understood. The easier problems may be done mechanically or from the knowledge of some formula.
9. Word problems are probably difficult. However, one should notice that most of the word problems in the test involve the idea of variation and proportion.
10. The exponential notation, for other than positive integral exponents, is not well known.

These statements are not positive because of the small number of questions on the test involving each mathematical concept. On the other hand, they give indications of what should be taught in collegiate courses. They also show a method of classification of questions which could be used for composite scoring in a more elaborate testing program. That is, all the questions involving fractions, for example, could be scored. All those involving quadratic equations could be scored, and so forth. On this basis, with reliable and adequate tests, positive results should follow.

Since the method of construction of this test involves neither an elaborate theory nor much labor compared with the results obtained, similar tests may be constructed easily for use in a variety of similar situations. In conclusion, a copy of a part of the test in its present form is given.

1. Divide 10.75 by .005.

Perform the indicated operations: (questions 2-5)

2. $2/3+3/8=.....$
6. If a is a number between zero and one, which number is greater, a or a^2?
8. If $y=2x+6$, what value of x will make $y=0$?

Perform the indicated operations: (questions 11, 12)

11. $\sqrt{.0049}=.....$
14. Solve for t, $8=3(2+at)$.
15. If x is positive, does $x-2x$ have a positive or a negative value?

Simplify: (questions 17, 18)

17. $\sqrt[3]{c^3}=.....$

Perform the indicated operations: (question 19)

19. $\dfrac{a}{c} - \dfrac{2}{c} = ...$

Factor: (questions 20-23)

22. $a^2-10ab+25b^2$.

Find the value of: (questions 24-26)

25. b^2-4ac if $a=0$, $b=-2$, $c=-3$.

Are the following true or false? (questions 27-30)

28. $\sqrt{p+q}=\sqrt{p}+\sqrt{q}$.
31. What should be the value of K so that $x=2$ will satisfy the equation
$$x^2-6x+K=0 ?$$
33. Find the value of x which satisfies the following equation:
$$\frac{x}{3-x} = 5.$$
37. If one of two equal base angles of a triangle is 40 degrees, how many degrees is the vertex angle?
39. Triangles ABC and $A'B'C'$ are similar. Side $AB=12$ inches. Side $BC=18$ inches. Side $A'B'=16$ inches. How long is side $B'C'$?
40. From a square sheet of tin of side 10 inches, four squares are removed, each of side 3 inches. What is the area of the tin left in the original sheet?

41. The area of a circle is πr^2. If the radius of a circle is increased from k inches to $5k$ inches, how many times as large will the area become?

43. The terms in the given sequence follow a certain law. Continue with one term in each direction according to the same law. That is, fill in the missing numbers. ......,3,8,13,......

46. A trucking company has enough gasoline on hand to last 30 trucks for 10 days. If ten trucks are sold, how long would the gasoline last the remaining 20 trucks?

47. A substance is composed of molecules each of which occupies a volume of 2×10^{-8} cubic centimeters. How many molecules are there in 5 cubic centimeters of the substance?

48. Simplify: $\dfrac{\sqrt{5x}}{\sqrt{125x}} = \ldots\ldots$

49. Perform the indicated operations: $\dfrac{2}{x^3} - \dfrac{5}{3x} = \ldots\ldots$

50. Solve for x and y : $2x+4y=-8$
$2x+y=1.$

53. Construct the graph of $y=2x-1$. On the graph label with P the point (3,5).

55. Solve for x : $x^2+.6x+.09=0$.

56. For what value of K is the expression x^2-5x+K equal to the product of two factors that are alike.

58. A dry sponge weighs 2 ounces. It is then soaked with water. If 80 percent of the total weight of the wet sponge is water, how much will the wet sponge weigh?

59. If $10^{0.4771}=3$, find the value of $10^{4.4771}$.

60. Find x if $(81)^x=3$.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, Mathematics, L. S. U., Baton Rouge, Louisiana.

SOLUTIONS

No. 420. Proposed by *Nelson Robinson,* Louisiana State University.

Let

$$A_{ij}^{(n)}=\sum_k(-1)^k\frac{i!j!(n-k)!}{n!(i-k)!(j-k)!k!}\qquad \begin{array}{c}k=0,\cdots,j;\ k\leqq i\leqq j,\\ i,j=0,\cdots,n,\end{array}$$

$$B_{ij}^{(n)}=\frac{(n-i)!(n-j)!}{(n-i-j)!n!}\qquad n\geqq i+j,$$

$$B_{ij}^{(n)}=0\qquad i\leqq n<i+j.$$

Prove $$A_{ij}^{(n)}=B_{ij}^{(n)}.$$

Solution by the *Proposer.*

Let $j=0,\cdots,n$ and for the moment suppose $i<j$. We may write

$$(1)\qquad A_{ij}^{(n)}=\sum_{k=0}^{i}(-1)^k\frac{{}_iC_k\cdot{}_jC_k}{{}_nC_k}$$

$$(2)\qquad A_{i+1,j}^{(n)}=\sum_{k=0}^{i}(-1)^k\frac{{}_{i+1}C_k\cdot{}_jC_k}{{}_nC_k}+(-1)^{i+1}\frac{{}_jC_{i+1}\cdot{}_iC_i}{{}_nC_{i+1}},$$

where in the last term we have put ${}_iC_i$ for ${}_{i+1}C_{i+1}=1$. Substituting for ${}_{i+1}C_k$ its value from

$${}_{i+1}C_k={}_iC_k+{}_iC_{k-1},$$

and noting that when $k=0$, ${}_iC_{k-1}=0$, we have the following reductions,

$$A_{i+1,j}{}^{(n)}=\sum_{k=0}^{i}(-1)^k\frac{{}_iC_k\cdot{}_jC_k}{{}_nC_k}+\sum_{1+k=0}^{i}(-1)^{k+1}\frac{{}_iC_k\cdot{}_jC_{k+1}}{{}_nC_{k+1}}$$

$$+(-1)^{i+1}\frac{{}_iC_i\cdot{}_jC_{i+1}}{{}_nC_{i+1}},$$

$$=A_{ij}{}^{(n)}+\sum_{k=0}^{i}(-1)^{k+1}\frac{{}_iC_k\cdot{}_jC_{k+1}}{{}_nC_{k+1}},$$

$$=A_{ij}{}^{(n)}-\frac{j}{n}\sum_{k=0}^{i}(-1)^k\frac{{}_iC_k\cdot{}_{j-1}C_k}{{}_{n-1}C_k},$$

(3) $$A_{i+1,j}{}^{(n)}=A_{ij}{}^{(n)}-\frac{j}{n}A_{i,j-1}{}^{(n-1)}.$$

Taking ${}_jC_{j+1}={}_jC_{j+2}={}_jC_{j+3}=\cdots=0$, we see that the above holds also for $i=j$, and in fact for $i>j$.

Now the proposed equation

(4) $$A_{ij}{}^{(n)}=B_{ij}{}^{(n)}$$

may be easily verified for $i=0$ and all j and n. In order to prove (4) by induction on i it is sufficient to show that

$$A_{i+1,j}{}^{(n)}=\frac{(n-i-1)!(n-j)!}{(n-i-j-1)!n!},\qquad i<n-j;$$

$$A_{i+1,j}{}^{(n)}=0,\qquad i\geqq n-j$$

assuming that

(5) $$A_{ij}{}^{(n)}=\frac{(n-i)!(n-j)!}{(n-i-j)!n!},\quad i\leqq n-j;\quad A_{ij}{}^{(n)}=0,\quad i>n-j,$$

for all values of j and n. But from equations (5) and (3) we have

$$A_{i+1,j}{}^{(n)}=\frac{(n-i)!(n-j)!}{(n-i-j)!n!}-\frac{j}{n}\cdot\frac{(n-i-1)!(n-j)!}{(n-i-j)!(n-1)!}$$

$$=\frac{(n-i-1)!(n-j)!}{(n-i-j-1)!n!},\quad i<n-j;$$

$$A_{i+1,j}{}^{(n)}=\frac{j!(n-j)!}{n!}-\frac{j}{n}\cdot\frac{(j-1)!(n-j)!}{(n-1)!}=0,\quad i=n-j;$$

$$A_{i-1,j}^{(n)}=0 - \frac{j}{n}\,0=0, \qquad i>n-j.$$

Thus the induction is complete and (4) is established.

No. 431. Proposed by *E. P. Starke*, Rutgers University.

Show that it is possible to find a number such that, if its digits be written down in order twice, a perfect square is formed. What is the smallest possible number of digits?

Solution by the *Proposer*.

Let the desired number have n digits, $abcd\cdots$, so that the hypothesis may be expressed as

$$abcd\cdots abcd\cdots=(abcd\cdots)(10^n+1)=x^2,$$

where $abcd\cdots<10^n$. This equation is evidently impossible unless 10^n+1 is divisible by a square greater than 1. From the elementary theory of congruences it is easy to show that 10^n+1 is never divisible by 3, is divisible by 7^2 only if n is an odd multiple of 21, is divisible by 11^2 only if n is an odd multiple of 11, by 13^2 for n an odd multiple of 39, etc. It thus appears probable that 11 is the smallest value of n. With this value we have

$$x^2=(10^{11}+1)(abcd\cdots)=11^2\cdot 826446281(abcd\cdots).$$

The right member will be a square if $abcd\cdots$ is 826446281 or a multiple by a square factor. We choose this factor 16, 25, $\cdots$, 100, of such size as to produce an 11-digit number. The smallest resulting value is thus

$$1322314049613223140496=(36363636364)^2.$$

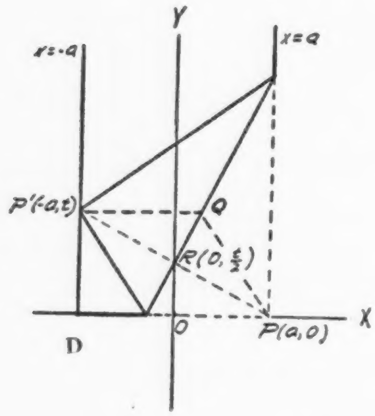
No. 448. Proposed by *M. S. Robertson*, Rutgers University.

A long rectangular strip of paper has one corner folded over so as just to touch the opposite side. Show that the creases formed in this way envelope an arc of a parabola whose directrix and latus rectum produced are the two long sides of the rectangular strip.

Solution by *C. N. Mills*, Normal, Illinois.

Let $2a$ denote the width of the strip. Choose X and Y axes as shown. Fold P so as to fall at the point $P'(-a,t)$ on the left side of the strip. The equation of the crease is (with parameter t):

$$f(x,y,t)\equiv 2ty-t^2-4ax=0.$$



Differentiate with respect to t. We find $t=y$ and hence the equation of the envelope is

$$y^2=4ax,$$

a parabola with directrix $x=-a$ and focus P.

Also solved by *P. D. Thomas* who supplied the figure given here.

Editor's Note: Since the perpendicular bisector of PP' is the locus of points equidistant from P and P', it contains a point Q such that $QP'(=QP)$ is perpendicular to DP'. Thus, since the crease bisects angle $P'QP$ and thus insures the optical property, all creases are tangents at points Q to a parabola with directrix DP' and focus P.*

No. 449. Proposed by *Paul D. Thomas*, Lucedale, Mississippi.

A variable circle is tangent to one of two perpendicular lines and intercepts a chord of given length on the other. Find the locus of the center.

Solution by *Ray Hassler*, undergraduate, University of Oklahoma.

Let the tangent line be the Y-axis and let the line on which the circle intercepts a given chord, $2c$, be the X-axis. Then if the center of the circle be (x, y), the radius will always be x, and the perpendicular to the X-axis, y, will bisect the chord. From the right triangle formed by this ordinate, y, a radius, x, and the semi-chord, c, we can immediately write the locus of the center, $x^2-y^2=c^2$, which is a hyperbola.

Also solved by *W. H. Bradford, M. I. Chernofsky, W. B. Clarke, J. M. Hurt, L. Shenfil*, and the *Proposer*.

*See Yates, R. C.: *Tools, A Mathematical Sketch and Model Book* (1941) 58.

The following problem has already been solved, this volume, page 356. However, a solution in rhyme is of interest.:

No. 450. Proposed by *Alice M. H'Doubler*, student, Bryn Mawr College.

A certain youth was asked his age
By one who seemed to be a sage;
To whom the youth made this reply,
Sir, if you wish your skill to try,
Eight times my age increased by four
A perfect square, nor less nor more;
Its triple square plus nine must be
Another square as you will see.
He tried but sure it posed him quite,
His answer being far from right.
You skilled in science I implore
This mystic number to explore.

Solution by *Robert V. Sweeney*, student, Colgate University.

Oh youth, to you I do reply,
For here I wish my skill to try.
At first I take your age to be
Some integer unknown to me.
The first condition, as we explore,
(If each be multiplied by four)
By figurate numbers of order three
Is satisfied at once you see.

The second condition is named after Pell—
Its integer solutions are easy to tell;
They have a law of recurrence as well.
First comes 0 and next 'tis 3;
If you multiply this last by four
And subtract at once the one before—
Yes, this goes on forevermore!

The first gives 0, 4, 12 and 24,. . .
The second gives 0, 3, 12 and many more.
Since younger than Methuselah, youth, thou must be,
Then twelve is the age in years for thee.

No. 451. Proposed by *N. A. Court*, University of Oklahoma.

Two positive segments PP', QQ' move, in any manner whatever, and independently of one another, on two fixed skew lines. The points L, M divide the segments PQ, $P'Q'$ in a given ratio, both in magnitude and in sign. Show that the direction of the line LM and the length of the segment LM are fixed.

Solution by *Clarence M. Ablow*, Los Angeles, California.

Let P, Q, P', Q', L, M be vectors from an origin to the respective points. The conditions of the problem are expressed vectorially by:

$$P-P'=\text{constant}, \qquad Q-Q'=\text{constant},$$

$$L=P+k(P-Q) \qquad M=P'+k(P'-Q'),$$

where k is the given ratio. It follows directly that

$$L-M=(P-P')+k[(P-P')-(Q-Q')]$$

is constant so that the length and direction of the line segment LM are constant.

Also solved by the *Proposer*, who notes that the corresponding property in the plane was considered in Educational Times, Reprints, Vol. 64 (1896), p. 31, Q. 12761.

No. 454. Proposed by *James F. Callicott*, student, Colgate University.

Assuming that the curve $y=x^3+ax^2+bx+c$ has a maximum point at A and a minimum point at B, show that the points on the curve where the curvature is greatest are located outside that part of the curve which is between A and B.

Solution by *W. V. Parker*, Louisiana State University.

By translating axes the equation of the curve may be written as $y=x^3-3p^2x$. Its maximum point is at $(-p, 2p^3)$ and its mimimum point is at $(p, -2p^3)$, $p>0$. The curvature at any point is

$$K=6x/(9x^4-18p^2x^2+9p^4+1)^{3/2}.$$

The abscissas of points for which $dK/dx=0$ are roots of

$$45x^4-36p^2x^2-9p^4-1=0,$$

or

$$x^2=[6p^2+\sqrt{(81p^4+5)}]/15.$$

Since this last quantity is greater than p^2, the points of greatest curvature lie outside that portion of the curve between its maximum and minimum points.

Also solved by the *Proposer*.

No. 456. Proposed by *V. Thébault*, Tennie (sarthe-France).

In an orthocentric tetrahedron $ABCD$, inscribed in a sphere (O), the first twelve point sphere is (O_1) and the second twelve point sphere

(O_2). Show that the lengths of the tangents to (O_1) and (O_2), drawn from an arbitrary point of (O), are in the ratio of $\sqrt{3}$ to 2.

Solution by *Paul D. Thomas*, Lucedale, Mississippi.

If H is the orthocenter of the orthocentric tetrahedron, then O_1 is the midpoint of OH, and O_2 divides OH internally in the ratio 1:2. (The Euler line).* Also (O), (O_1), (O_2) are coaxal.†

If d is the distance from an arbitrary point, R, of (O) to the common radical plane of (O), (O_1), (O_2), then the power of R with respect to (O_1) is $p_1 = 2d\cdot\overline{OO_1} = 2d\cdot\overline{OH}/2 = d\cdot\overline{OH}$.‡ Similarly the power of R with respect to (O_2) is

$$p_2 = 2d\cdot\overline{OO_2} = 2d\cdot 2\,\overline{OH}/3 = 4d\cdot\overline{OH}/3.$$

Thus $p_1/p_2 = \frac{3}{4}$, and this is the ratio of the squares of the tangents.

PROPOSALS

No. 472. Proposed by *Paul D. Thomas*, Lucedale, Miss.

If a, b, c, d are the lengths of the sides of a quadrilateral which admits both a circumscribed and inscribed circle, show that the inradius is

$$r = \frac{\sqrt{abcd}}{s},$$

where $2s$ is the perimeter.

No. 473. Proposed by *Howard D. Grossman*, New York City.

Prove the obvious generalization of the following relation:

$$\sum_{x=1}^{n} x^5 = \frac{n(n+1)}{6}\begin{vmatrix} 2 & 0 & 0 & 0 & 1 \\ -1 & 3 & 0 & 0 & n \\ 1 & -3 & 4 & 0 & n^2 \\ -1 & 4 & -6 & 5 & n^3 \\ 1 & 15 & 10 & -10 & n^4 \end{vmatrix},$$

where the portion of the determinant below the principal diagonal is identical with a portion of the Pascal triangle except for the negative signs in alternate diagonals. The determinant is unchanged in value if all signs are made positive and n is replaced by $n+1$.

**Modern Pure Solid Geometry*, Nathan Altshiller Court, p. 262.
†*Ibid.*, p. 264.
‡*Ibid.* p. 184.

No. 474. Proposed by *W. V. Parker,* Louisiana State University.

There is one, and only one, triangle of maximum area inscribed to the ellipse $x^2/a^2+y^2/b^2=1$, with vertex at any point P of the ellipse. Find the locus of the circumcenter of this triangle as P moves around the ellipse.

No. 475. Proposed by *E. P. Starke,* Rutgers University.

The triangle formed by three tangents to the ellipse, $x=a\cos\Theta$, $y=b\sin\Theta$, has the area

$$S=ab\tan\frac{\Theta_2-\Theta_1}{2}\tan\frac{\Theta_3-\Theta_2}{2}\tan\frac{\Theta_1-\Theta_3}{2},$$

where $0\leqq\Theta_1<\Theta_2<\Theta_3<2\pi$ are the values of the parameter for the three points of contact. The value of S is positive or negative according as the ellipse is inscribed in the triangle or not. The formula for the area of the triangle inscribed in the ellipse with vertices at these points of contact is

$$2ab\sin\frac{\Theta_2-\Theta_1}{2}\sin\frac{\Theta_3-\Theta_2}{2}\sin\frac{\Theta_3-\Theta_1}{2}.$$

No. 476. Proposed by *Nelson Robinson,* Louisiana State University.

The length of one arch of a hypocycloid is equal to the subtended arc of the fixed circle upon which the curve is generated. What is the relation between the radii of the fixed and rolling circles?

No. 477. Proposed by *D. L. MacKay,* Evander Childs High School, New York.

Given points A and B. Draw the line AB with a straightedge whose length is less than $(AB)/2$.

No. 478. Proposed by *Howard D. Grossman,* New York City.

If on each side of any triangle I as base, an equilateral triangle be constructed outwardly, the centroids of these three equilateral triangles themselves form an equilateral triangle II. If the original equilateral triangles be constructed inwardly, their centroids form another equilateral triangle III. Prove that the difference of areas of II and III is equal to the area of I.

Bibliography and Reviews

Edited by
H. A. SIMMONS and JOHN W. CELL

Fundamental Mathematics. By Duncan Harkin. Prentice-Hall, Inc., New York, 1940. xv+434 pages. $3.00.

"There are many who would like to know the very human story of mathematics without going through the often stultifying drudgery of the standard curriculum, to understand its principles in outline without specializing in technical difficulties, and still get enough technique in the process to handle the simpler parts with some ease and confidence." In this foreword the author sets forth the fundamental purpose of the book.

The book begins with a development of number concept and ends with simple applications of integral calculus. It contains a great variety of subjects from Arithmetic, Algebra, Trigonometry, Analytic Geometry, and the Calculus; e. g., magic squares, modular arithmetic, fractions, binomial theorem, arithmetic series, mathematical induction, solution of algebraic equations, proportion, continued fractions, similarity, circular functions, the differential operator, Taylor Series, integration, and others. The methods of presentation are interesting and, in accordance with the author's aim, not along traditional lines. In my opinion this is not an altogether happy innovation, particularly so in the chapter on the quadratic equation. The chapter on MacNeish-Desargues Finite Geometry,* devoted to problems involving only the joining of straight lines and points, is new. Throughout there is a large number of problems which stimulate the curiosity; many of them are practical. Interest is added to many topics and problems by giving their historical background. References to other writings are made freely, altogether two hundred eighty-one. The photographs and drawings, of which there are many, are very good.

Many persons with a normal amount of high school training in mathematics will be able to read the book without assistance though most freshmen in college will require the assistance of a careful teacher. A reader eager to get on with the mathematics may consider portions of the book irrelevant to the author's purpose.

University of Tennessee. J. A. COOLEY.

The Development of Mathematics. By E. T. Bell, McGraw-Hill Book Co., New York, London, 1940. xi+583 pages. Cloth, $5.00.

This book is designed to be a broad survey and historical analysis of the growth of mathematics; certain philosophical elaborations are also present. It is *not* a history in the usual sense of the term, but does suggest new connections between old ideas or new applications of old methods. It may provide the reader with a deeper grasp of the entire field. There is a laudable attempt to present mathematics within the frame-

*The author acknowledges his indebtedness to H. F. MacNeish of Brooklyn College for this section.

work of general history, covering the period from *circa* 4000 B. C. to 1940. About three-quarters of the book is devoted to mathematics after Newton; about half to mathematics since Gauss. Special attention is given to the influence of the sciences on the development of modern mathematics. Only main trends (decisive epochs) of the past six thousand years are taken up, and these are presented only through typical major episodes in each. An outstanding feature of the volume is the explanation of the various reasons why some topics continue to interest mathematicians and others are ignored or dismissed as being of merely antiquarian interest. Specialists in certain fields are likely to disagree with some of the author's conclusions on this last point.

Besides giving the leading ideas, the text frequently gives technical hints in elucidating the mathematics involved. To many readers the discussion of mathematical philosophy and mathematical logic will prove interesting and stimulate thought. The statistical method appears as social mathematics. The author discusses fully those epochs when standards of mathematical proof changed abruptly—the Fifth century B. C., the 1820s and the 1930s.

Footnotes have been kept at the irreducible minimum and placed at the back of the book with other bibliographical references. One can go too far in eliminating such notes. For instance, the reader will want to know where there is evidence supporting the assertion (p. 486) that Gauss considered Lamé the foremost French mathematician of his generation. There is a complete and useful index; mathematicians' initials and dates are given. The author has adopted the cautious device of indicating a man's nationality along with the dates. Occasionally there is some question here; Euler is usually listed as Swiss, although he did much work in other countries. Land of birth is one criterion. The physical make-up of the volume is splendid; misprints are not frequent for a study of this type.

These words are appropriate in closing a review of the work: "This has been an opportunity to do something a little off the beaten track to show prospective readers how the mathematics familiar to them got where it is, and where it is going from there. I trust that students will tolerate the departure from the traditional textbook. For one thing, at any rate, the more sensible should be grateful: only the most ingenious instructor could set an examination on the book."

It is not so easy to show where mathematics is going from where it is.

State Teachers College, La Crosse, Wisconsin. G. WALDO DUNNINGTON.

First Year College Mathematics. By Richtmeyer and Foust. Crofts and Company, New York, 1942. xi+461 pages.

This book includes most of the topics usually taken up in introductory courses in college algebra, trigonometry, and plane analytic geometry, and includes also an introduction to calculus. The first 130 pages are devoted primarily to algebra, with the material being very skillfully unified by use of the function concept together with the early introduction of the derivative and its subsequent use in studying linear functions, quadratic functions, polynomial functions, rational functions, and irrational functions, respectively. The derivative is introduced in an intuitive manner and precise definitions of a limit and of a derivative are not given. Integration is discussed also, but it is disposed of in eight pages and is applied only to area problems involving integration of polynomials and to rate problems.

Following the material on algebra the next 115 pages provide an adequate course in plane trigonometry and logarithms. A discussion of approximate computations precedes the work in numerical trigonometry. The trigonometric functions are defined

first in terms of acute angles and the general definitions are given later. The chapters on trigonometry are followed by a brief chapter on systems of linear equations in two and three unknowns. Second and third order determinants are taken up in this chapter with geometric applications left as exercises. Systems of equations involving quadratic equations are also treated briefly.

The next 108 pages provide a course in plane analytic geometry which, although brief, includes the essential topics. The remainder of the book is devoted to miscellaneous topics in algebra which would not find a logical place in the unified treatment of the earlier chapters. The particular topics taken up are progressions, permutations, combinations, probability, mathematical induction and the binomial theorem, complex numbers, and natural logarithms. No tables are included.

A few errors were observed, most of which are of a minor nature. For example, the illustration on page 85, section 52, makes use of the factor theorem rather than its converse. The same error occurs in the statement just before exercise 12 on page 86. In the discussion of conditional trigonometric equations on page 196 no mention is made of checking the solutions. In chapter 19, in common with many analytic geometry texts, the proofs lack completeness. For example, it is shown that every point on the parabola satisfies the equation y^2+4px, but it is not shown that every point which satisfies this equation lies on the parabola. On page 313 it should be pointed out that $a>c$ before writing equation (5), while on page 317 it should be pointed out that $a<c$ before writing equation (9). On page 363 the statements in italics are incorrect, since they omit the word "constant."

There is an excellent selection of problems of the drill variety and also of the type to encourage independent thinking on the part of the student. A "*self-test*" is inserted at the end of each chapter, thus enabling the student to check his mastery of the material. To the present reviewer this appears to be an excellent pedagogical device. Answers to the *self-tests* are given in the back of the book.

The book provides sufficient material for a year course for students who have had three semesters of high school algebra but no trigonometry. It is also well suited to students who have had but one year of high school algebra, since there is a considerable amount of review material in elementary algebra.

The typographical and general arrangement of the book are good, and only a few misprints were observed.

University of Wisconsin. H. P. EVANS.

Harmonic Integrals. By W. V. D. Hodge. Cambridge University Press, Cambridge, England, 1941. ix+281 pp.

The ideas developed in this stimulating book are essentially an outgrowth of the theory of Abelian integrals on algebraic curves. This theory, which proved to be extremely useful in the investigation of birationally invariant properties of curves, was soon generalized to algebraic varieties of higher dimensions. Many of the problems arising in this generalization have not yet been solved, and it was while working on some of these that Hodge developed his more general treatment of integrals.

A harmonic integral, or, rather, the associated harmonic tensor, is a generalization of the familiar harmonic function of two- or three-dimensional Euclidean space. The general harmonic integral is defined over a much more general type of space known as a Riemannian manifold, and the first chapter of Hodge's book is devoted to a concise account of the topology and differential geometry of these spaces. This treatment is self-contained, and provides an excellent review for a person who has some acquaintance

with these subjects. A reader totally unfamiliar with them would probably have some difficulty in supplying all the details of the arguments. However, references are given to standard treatments.

In Chapter II the integral associated with a skew-symmetric tensor is introduced, and is shown to have close connections with the topology of the manifold. The harmonic integrals, discussed in the next chapter, are associated with a special type of tensor, defined by certain differential equations. A simple example of a harmonic integral is $\int Pdx - Qdy$, where P and Q are the real and the imaginary parts of an analytic function of $x+iy$. In this case the equations defining a harmonic integral are just the Cauchy-Riemann equations,

$$\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} = 0, \qquad \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = 0.$$

The basic property of harmonic integrals is expressed in the following theorem: If Γ_i, $i=1,\cdots,R_p$, is a basis for the p-cycles of a Riemannian manifold M, and if ν_i are arbitrary real numbers, then there is a unique harmonic p-fold integral on M whose integral over Γ_i is ν_i.

The remaining two chapters of the book are devoted to the properties of harmonic integrals on special types of Riemannian manifolds; namely, those associated with algebraic varieties and with continuous groups. In each of these cases the connection between the harmonic integrals and the topology of the manifold is worked out in some detail. The results are used to classify and determine the harmonic integrals on these manifolds, and also to show the close connection of harmonic integrals with certain previously known concepts. They also throw light on earlier theorems and point the way to extensions and elaborations of these.

Such a brief sketch of the contents of the book cannot serve to indicate the importance of Hodge's work. The ideas and the techniques developed here are already beginning to influence the work of other mathematicians, and it seems very likely that they will have a significant effect on the future development of topology and differential geometry, especially in the fields where these subjects overlap. Hence for a person interested in modern developments in geometry this book is practically required reading.

Cornell University. R. J. WALKER.

Mary Somerville. By A. W. Richeson. Scripta Mathematica, Vol. VIII, No. 1. March, 1941.

In this article is a short account of the life of Mary Somerville, an English scientist and writer who was active in the first half of the nineteenth century. Her story is of interest to mathematicians because she wrote the English translation of Laplace's *Mecanique Celeste*, at the same time rewriting and simplifying it so that it could be more readily understood than the original. Mrs. Somerville's story would appeal to most of us, however, mainly because it describes the difficulties of a woman's obtaining an education at that time.

A critical evaluation of the importance of Mrs. Somerville's work would be an ambitious undertaking, but one wishes that Professor Richeson had gone into it more thoroughly. Apparently very little of her work was original, though her contribution to the progress of science, in the form of the books *The Mechanisms of the Heavens*, and *Physical Geography*, was undoubtedly greater than that of many men whose work was "original." As far as her own life is concerned, Professor Richeson gives us most of the

pertinent facts contained in *The Personal Recollections of Mary Somerville*, a book which he fails to mention in his article. The book itself is rambling and varies greatly in interest, which is not surprising since Mrs. Somerville wrote it only shortly before she died, at the age of 92; but it is well worth looking into for the *pictures it gives* of the age in which she lived, of the great scientists of that age, among them her close friends Herschel and Laplace, and finally of that very attractive person, Mary Somerville herself. The book, unfortunately, contains very little of the scientific side of her life, possibly because it was edited by her daughter who seems to have been influenced by the belief, still held by most people, that the kindest thing one can say to a woman mathematician is that she does not look like one. Thus Mrs. Somerville's opinions and correspondence on scientific matters are suppressed in favor of her descriptions of Italian scenery.

Professor Richeson could not give much more than bare facts in his short article, but it is to be hoped that what he gave will stimulate an interest in the history of mathematics and its allied sciences.

Vassar College. AUDREY WISHARD.

Intermediate Algebra for College Students. By Thurman S. Peterson. Harper & Brothers, New York. 1942. viii+358 pages.

The very existence of this book is an admission of a need which most teachers of first year college mathematics have wanted to deny, knowing while they did so that a very large part of their students failed to master college algebra because of an inadequate foundation. The author states in his preface, "This book is designed to serve as a text for college students who have had not more than one year of secondary algebra. The objectives of the book are two-fold: in the first place, to serve as a terminal course in algebra preparing for non-scientific studies; and secondly, to serve as a foundation course in algebra preparing for more advanced college mathematics." This course does not displace the regular course in college algebra.

Many features of the book are noteworthy. The one which most impresses the reviewer is the constant emphasis upon certain fundamental concepts which are as essential to other divisions of mathematics as to algebra, such as: the meaning and varied applications of the fundamental principle of fractions; interpreting mathematically (and in algebra, symbolically) the "stated problem"; the universality of the linear relationship, $ax=b$; and the balance of equations. One might call the text somewhat inconsistent in this latter point in that considerable stress is given to the mechanics of "transposition", to the neglect of the actual operation of addition to or subtraction from, each member of an equation.

In his effort to simplify the processes, the author has not rigidly defined certain terms. He states, for instance, that the quadratic equation whose graph is tangent to the x-axis has only one solution and that a point of tangency of two graphs yields only one pair of common values of the variables of the equations. Again he states that the "sum" of an infinite geometric series is

$$S=\frac{a}{1-r} .$$

The phrase *limiting value* is found once in the context, but the student is given no notion of the concept of *limit*.

The text devotes twelve chapters (267 pages) to algebra through simultaneous quadratics, and gives only fifty-five pages to ratio, variation, the binomial theorem,

logarithms, and progressions in their simplest applications. The relative emphasis upon the various topics is in keeping with its stated purpose. Each sub-topic is explained fully and is accompanied by an excellent list of exercises. Another generous set of problems at the end of each chapter reviews the entire content of the chapter.

The reviewer has not found more valuable problems in any comparable book. Much opportunity is given in these problems to practice the handling of common and decimal fractions through their use as coefficients and constants; to evaluate symbolic expressions and to apply formulas (actual formulas) to specific data—all essential equipment of any informed person of whatever interests. Another strong point is the simple, clear, and fully illustrated discussion of the equations and graphs of the conic sections. Still another is the insistence throughout that solutions be checked for accuracy, or, as in the case of logarithms, estimated for reasonableness.

The book as a whole is excellently written and attractively arranged. Important principles, rules, and processes are brought into relief by bold-face type and barred enclosures. The very printing should give the student a confidence that here is a book which he can read and comprehend.

Western Kentucky State Teachers College. TRYPHENA HOWARD.

Fourier Series and Boundary Value Problems. By Ruel V. Churchill. McGraw-Hill Book Company, New York and London, 1941. ix+206 pages. $2.50.

This book is a very good introduction to Fourier Series and their application to the solution of boundary value problems in the partial differential equations of physics and engineering. The treatment is kept on an elementary level, so that the book can be well recommended as textbook, or as reading for anyone who is familiar with the elements of advanced calculus and ordinary differential equations. Chapter I is an introduction. Chapter II deals with some classical problems of physics. Chapter III discusses orthogonal sets of functions. Chapters IV and V are devoted to the elementary theory of Fourier Series and integrals. Chapter VI gives applications of the theory to boundary value problems. Chapter VII gives an account of the uniqueness problem. The last two chapters discuss the fundamental properties of Bessel functions and Legendre polynomials. At the end of each chapter numerous problems with answers elucidate the application of the different methods. A subject index facilitates the use of the book. The presentation is clear and makes pleasant reading. The author succeeds in presenting well selected material with clear exposition in a relatively small space. The book is a real contribution in textbook literature; it deserves a wide circulation.

University of Cincinnati. OTTO SZÁSZ.

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